Abstract

In this note, I collect some facts about derivative rules which can be derived using simple facts.

Keywords: Power rule, product rule, quotient rule
1 Derivative

Recall derivative $f'(x)$ of a function $f(x)$ is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided that the above limit exists. Also, recall that if $f'(x)$ exists, then $f(x)$ is continuous at $x$. That is,

$$\lim_{h \to 0} f(x + h) = f(x).$$

The following fact is well-known.

**Theorem 1** (*Geometric series*)

$$(1 - r) \left( 1 + r + r^2 + \ldots + r^{m-1} + r^m \right) = 1 - r^{m+1}.$$

**Proof:** We have

$$(1 - r) \left( 1 + r + r^2 + \ldots + r^{m-1} + r^m \right) = \left( 1 + r + r^2 + \ldots + r^{m-1} + r^m \right) - r \left( 1 + r + r^2 + \ldots + r^{m-1} + r^m \right)$$

$$= 1 - r^{m+1}.$$

More generally, we have the following theorem

**Theorem 2**

$$(a - b) \left( a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1} \right) = a^n - b^n.$$

**Proof:** We have

$$a^n - b^n = a^n \left( 1 - \left( \frac{b}{a} \right)^n \right)$$

$$= (1 - \frac{b}{a}) a^n \left( 1 + \frac{b}{a} + \left( \frac{b}{a} \right)^2 + \ldots + \left( \frac{b}{a} \right)^{n-1} \right)$$

$$= (a - b) a^{n-1} \left( 1 + \frac{b}{a} + \left( \frac{b}{a} \right)^2 + \ldots + \left( \frac{b}{a} \right)^{n-1} \right)$$

$$= (a - b) \left( a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1} \right).$$

**Remark 1** *Note that to get to result in Theorem 2, one can start from*

$$(a - b) \left( a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1} \right)$$

*and expand it. Terms will cancel out nicely but it will not natural to see where the term*

$$(a - b) \left( a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1} \right)$$

*comes from.*
Now we are ready for the power rule, which is one of the most useful derivative rules.

**Theorem 3** *(Power rule)*

\[(x^n)' = nx^{n-1}.\]

**Proof:**

\[
(x^n)' = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h} \\
= \lim_{h \to 0} \frac{(x + h - x)}{h} \left[ (x + h)^{n-1} + (x + h)^{n-2}x + \ldots + (x + h)x^{n-2} + x^{n-1} \right] \\
= \lim_{h \to 0} \left[ (x + h)^{n-1} + (x + h)^{n-2}x + \ldots + (x + h)x^{n-2} + x^{n-1} \right] \\
= nx^{n-1}.
\]

**Remark 2** *In the classic calculus text book [Stewart (2012)](https://www.stewartcalculus.com), the power rule is derived using (implicit) binomial expansion, which is not natural since most calculus students may have not seen it before.*

To derive the product rule and the quotient rule, we need the following simple facts:

**Theorem 4** \((f^2(x))' = 2f(x)f'(x).\)

**Proof:** Note that

\[
(f^2(x))' = \lim_{h \to 0} \frac{(f(x + h))^2 - (f(x))^2}{h} \\
= \lim_{h \to 0} \left( f(x + h) + f(x) \right) \frac{(f(x + h) - f(x))}{h} \\
= \lim_{h \to 0} (f(x + h) + f(x)) \lim_{h \to 0} \frac{(f(x + h) - f(x))}{h} \\
= 2f(x)f'(x).
\]

**Theorem 5** \(\left( \frac{1}{f(x)} \right)' = -\frac{f'(x)}{f^2(x)}.\)

**Proof:** Note that

\[
\left( \frac{1}{f(x)} \right)' = \lim_{h \to 0} \left( \frac{1}{f(x + h)} - \frac{1}{f(x)} \right) \frac{1}{h} \\
= \lim_{h \to 0} \left( \frac{f(x) - f(x + h)}{h} \right) \frac{1}{f(x)f(x + h)} \\
= \lim_{h \to 0} \left( \frac{f(x)}{h} \right) \lim_{h \to 0} \frac{1}{f(x)f(x + h)} \\
= -\frac{f'(x)}{f^2(x)}.
\]

Now we are ready to prove the product rule and quotient rule.
Theorem 6 (Product rule)

\[(fg)' = f'g + fg'.\]

**Proof:** On one hand, using the Theorem 4 we have

\[
[(f + g)^2]' = 2(f + g)(f' + g')
= 2(ff' + fg' + f'g + gg').
\]

On the other hand, we have

\[
[(f + g)^2]' = (f^2 + 2fg + g^2)'
= 2(ff' + (fg)' + gg').
\]

From (2) and (3), it is immediate to see that

\[(fg)' = f'g + fg'.\]

Theorem 7 (Quotient rule)

\[
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}
\]

**Proof:** This is a direct application of the product rule and Theorem 5. Concretely, we have

\[
\left(\frac{f}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)'
= f'\frac{1}{g} + f\left(-\frac{g'}{g^2}\right)
= \frac{f'g - fg'}{g^2}.
\]

Remark 3 In [Stewart (2012)], the product rule and the quotient rule are derived using the definition of derivative, which is fine but with some heavy calculations. The derivations presented here clearly require less work.

References